

# Symmetry problem 1

Alexander G. Ramm

Mathematics Department, Kansas State University,

Manhattan, KS 66506-2602, USA

[ramm@math.ksu.edu](mailto:ramm@math.ksu.edu)

<http://www.math.ksu.edu/~ramm>

## Abstract

A symmetry problem is solved. A new method is used. The idea of this method is to reduce to a contradiction the PDE and the over-determined boundary data on the boundary.

The new method allows one to solve other symmetry problems.

## 1 Introduction

Symmetry problems for PDE were studied in many publications by many authors, see, for example, [1]. In this paper a new method is given for a study of symmetry problems for PDE. Throughout we assume that  $D$  is a bounded connected  $C^2$ -smooth domain in  $\mathbb{R}^3$ ,  $S$  is the boundary of  $D$ ,  $N$  is the unit normal to  $S$ , pointing out of  $D$ ,  $u_N$  is the normal derivative of  $u$  on  $S$ ,  $D' = \mathbb{R}^3 \setminus D$ ,  $S^2$  is the unit sphere in  $\mathbb{R}^3$ ,  $J_n(r)$  is the Bessel function regular at  $r = 0$ ,  $j_\ell(r)$  is the spherical Bessel function,  $j'_\ell(kr) = \frac{dj_\ell(kr)}{dr}$ ,  $k > 0$  is a constant,  $\beta \cdot y = (\beta, y)$  is the dot product.

In [2]–[10] the author studied various symmetry problems.

Let us formulate the symmetry problem studied in this paper. Our main result is formulated in Theorem 1.

**Theorem 1.** *Assume that*

$$\Delta u + k^2 u = 0 \quad \text{in } D, \quad u|_S = 1, \quad u_N = 0. \quad (1)$$

*Then  $S$  is a sphere of radius  $a$  where  $a$  solves the equation  $j'_0(ka) = 0$ .*

In [5], [7] it was shown that the Pompeiu problem is equivalent to the problem (1).

In Section 2 proofs are given.

---

MSC: 35J05, 35R30

Key words: symmetry problems for PDE



## 2 Proofs

*Proof of Theorem 1.* Let  $g(x, y, k) := \frac{e^{ik|x-y|}}{4\pi|x-y|}$ . If problem (1) has a solution then this solution is unique by the uniqueness of the solution to the Cauchy problem for elliptic equation (1). The solution to equation (1) by Green's formula is:

$$u(x) = - \int_S g_N(x, t) dt, \quad x \in D; \quad u(x) = - \int_S g_N(x, t) dt = 0, \quad x \in D'. \quad (2)$$

Let  $B_R = \{x : |x| \leq R\}$ ,  $D \subset B_R$ . If  $D$  is a ball  $B_a$  of radius  $a$  and  $j'_0(ka) = 0$ , then problem (1) in  $B_a$  has a solution:

$$u = \frac{j_0(kr)}{j_0(ka)}, \quad r = |x|. \quad (3)$$

In what follows we assume that  $D \subset \mathbb{R}^2$  and  $S$  is a closed smooth curve. Let  $\mathbf{r}(s) = x(s)e_1 + y(s)e_2$  be a parametric representation of  $S$ ,  $s$  be the arc length along  $S$  and also the corresponding to the arc length  $s$  point on  $S$ ,  $\{e_1, e_2\}$  is a Cartesian basis in  $\mathbb{R}^2$ . The first boundary condition in (1) is  $u(x(s), y(s)) = 1$ . Differentiating with respect to  $s$  one gets  $u_x \dot{x} + u_y \dot{y} = 0$  and another differentiation yields

$$u_{xx} \dot{x}^2 + 2u_{xy} \dot{x} \dot{y} + u_{yy} \dot{y}^2 = 0, \quad \dot{x} = \frac{dx}{ds}. \quad (4)$$

Here we used the formula  $u_x \ddot{x} + u_y \ddot{y} = 0$ . This formula can be derived as follows:  $\nabla u \cdot \ddot{\mathbf{r}} = \nabla u \cdot \kappa \nu$ , where  $\kappa = \kappa(s) > 0$  is the curvature of  $S$ ,  $\nu = -N$  is the unit normal pointing into  $D$ ,  $\nabla u \cdot \nu = -u_N = 0$  on  $S$ . From (1) it follows that

$$u_{xx} + u_{yy} = -k^2 \quad \text{on } S. \quad (5)$$

Let us prove that (4) and (5) are not compatible at some points, except when  $S$  is a circle of radius  $a$ , where  $a$  solves the equation  $j'_0(ka) = 0$ .

Denote  $u_{xx} = p = p(s)$ ,  $u_{xy} = q = q(s)$ . Then (5) implies  $u_{yy} = -k^2 - p$  on  $S$ . Let  $A$  be a  $2 \times 2$  matrix with elements  $A_{11} = p$ ,  $A_{12} = A_{21} = q$ ,  $A_{22} = -k^2 - p$ . The equation for finding the eigenvalues  $\lambda_{1,2}$  of  $A$  is:

$$\lambda^2 + k^2 \lambda - p^2 - q^2 - k^2 p = 0. \quad (6)$$

The eigenvalues  $\lambda_1$  and  $\lambda_2$  are:

$$\lambda_{1,2} = -\frac{k^2}{2} \pm (k^4/4 + p^2 + q^2 + k^2 p)^{1/2}. \quad (7)$$

Clearly,  $\lambda_1 + \lambda_2 = -k^2$ ,  $\lambda_1 \lambda_2 = -p^2 - q^2 - k^2 p$ ,  $k^4/4 + p^2 + q^2 + k^2 p = (\frac{k^2}{2} + p)^2 + q^2 \geq 0$ . Thus,  $\lambda_2 < 0$ .

The corresponding eigenvectors (non-normalized but orthogonal) can be calculated explicitly. One has

$$e_1 = \{1, \gamma\}, \quad \gamma := \frac{q}{k^2 + p + \lambda_1} = \frac{\lambda_1 - p}{q}. \quad (8)$$

If  $q \neq 0$ , then

$$e_2 = \left\{ \frac{k^2 + p + \lambda_2}{q}, 1 \right\} = \{-\gamma, 1\}. \quad (9)$$

If  $q \neq 0$  then one checks that  $\frac{k^2 + p + \lambda_2}{q} = \frac{q}{\lambda_2 - p}$  and  $\frac{q}{\lambda_2 - p} + \frac{q}{k^2 + p + \lambda_1} = 0$ , so  $\gamma = -\frac{k^2 + p + \lambda_2}{q}$ .

If  $q = 0$  then  $\lambda_1 = p$ ,  $\lambda_2 = -p - k^2$ ,  $e_1 = \{1, 0\}$ ,  $e_2 = \{0, 1\}$ , and equation (13) (see below) leads also to a contradiction as in the case  $q \neq 0$ .

Clearly,  $e_1 \cdot e_2 = 0$ ,  $\|e_1\|^2 = \|e_2\|^2 = 1 + \gamma^2$ , so  $\gamma^2$  is invariant under rotations of the Cartesian coordinate system.

Denote  $\{\dot{x}, \dot{y}\} := w$ . Note that  $\dot{x}^2 + \dot{y}^2 = 1$ . Let  $c_1, c_2$  be scalar coefficients. Then

$$c_1 e_1 + c_2 e_2 = w, \quad w := \{\dot{x}, \dot{y}\}. \quad (10)$$

Solving explicitly this algebraic system for  $c_1$  and  $c_2$  one gets:

$$c_1 = (\dot{x} + \gamma \dot{y}) \Delta^{-1}, \quad \Delta = 1 + \gamma^2, \quad (11)$$

and

$$c_2 = (\dot{y} - \gamma \dot{x}) \Delta^{-1}. \quad (12)$$

Equation (4) can be written as  $(Aw, w) = 0$ . Substitute  $w$  from (10) into the equation  $(Aw, w) = 0$  and use the orthogonality of  $e_1$  and  $e_2$  to get

$$(\dot{y} - \gamma \dot{x})^2 \lambda_2 + (\dot{x} + \gamma \dot{y})^2 \lambda_1 = 0. \quad (13)$$

We now prove that (13) leads to a contradiction unless  $S$  is a circle of radius  $a$  where  $a$  solves the equation  $J'_0(ka) = 0$  if  $D \subset \mathbb{R}^2$  and  $a$  solves the equation  $j'_0(ka) = 0$  if  $D \subset \mathbb{R}^3$ .

Choose Cartesian coordinates in which  $\dot{x}(s) = -\gamma \dot{y}$ . Such coordinate system does exist because the only restriction on  $\dot{x}$  and  $\dot{y}$  is  $\dot{x}^2 + \dot{y}^2 = 1$  at all  $s \in S$ . Then, since  $\lambda_2 < 0$  equation (13) with  $\dot{x}(s) = -\gamma \dot{y}$  implies  $\dot{y}(1 + \gamma^2) = 0$ . Thus,  $\dot{y} = 0$ . Therefore,  $\dot{x} = \dot{y} = 0$ . This contradicts the relation  $\dot{x}^2 + \dot{y}^2 = 1$ . This contradiction holds for any smooth  $S$  except for a circle of a special radius, see (3).

Theorem 1 is proved.  $\square$

# References

- [1] T. Chatelain and A. Henrot, Some results about Schiffer's conjectures, *Inverse Problems*, 15, (1999), 647-658.
- [2] A. G. Ramm, *Inverse Problems*, Springer, New York, 2005.
- [3] A. G. Ramm, A symmetry problem, *Ann. Polon. Math.*, 92, (2007), 49-54.
- [4] A. G. Ramm, Symmetry problem, *Proc. Amer. Math. Soc.*, 141, N2,(2013), 515-521.
- [5] A. G. Ramm, The Pompeiu problem, *Global Journ. of Math. Analysis (GJMA)*, 1, N1, (2013), 1-10.  
Open access Journal:  
<http://www.sciencepubco.com/index.php/GJMA/issue/current>
- [6] A. G. Ramm, A symmetry result for strictly convex domains, *Analysis*, 35 (1), (2015), 29-32.
- [7] A. G. Ramm, Solution to the Pompeiu problem and the related symmetry problem, *Appl. Math. Lett.*, 63, (2017), 28-33.
- [8] A. G. Ramm, *Scattering by obstacles*, D.Reidel, Dordrecht, 1986.
- [9] A. G. Ramm, *Scattering by obstacles and potentials*, World Sci. Publ., Singapore, 2017.
- [10] A. G. Ramm, Necessary and sufficient condition for a surface to be a sphere, *Open J. Math. Anal.*, 2, (2018), issue 2, 51-52.  
Open access: <https://pisrt.org/psr-press/journals/oma/>